$\mu=\mu_0$ and $\alpha=\alpha_0=0$ (lossless resonator), this equation reduces to

$$-\frac{\tan\beta_0(l-d)}{\beta_0}=\frac{\tan\beta d}{\beta}, \qquad (4)$$

which is just (31) in Horner, *et al.*

Eq. (3) together with the relations $h^2-\gamma^2=s^2\mu\epsilon_c$ and $h^2-\gamma^2=-s^2\mu_0\epsilon_0$ impose six conditions on a total of eight field parameters, namely α_0, β_0, α, β, ω, ω_i, ϵ and σ. Solutions for any six of these may, therefore, be obtained provided that measurements have been made to evaluate the remaining two. It should be emphasized, however, that such measurements must be conducted while the sample-filled cavity is undergoing a transient response under the influence of a unit impulse of excitation. For it is then that the oscillations are natural. However, the field in the cavity may decay so rapidly that no measurements can be performed with any degree of accuracy. Of course, if the cavity losses are relatively small, as is often the case in practice, the necessary measurements can be made under steady-state sinusoidal operating conditions. It is obvious, therefore, that the equations of free oscillations are valid representations of forced oscillations only if the cavity is virtually lossless. A different set of equations is needed when the losses in the cavity are relatively high.

Careful examination of the conditions for free oscillations shows that, if the specimen under test is characterized by a finite, nonzero conductivity, the frequencies of free oscillations and the associated propagation constants are complex. The important implication of this fact is often overlooked. It is generally true that, when driven sinusoidally in time, the specimen-loaded cavity cannot be forced to oscillate at any one of its natural frequencies and that, therefore, (3) does not hold under these conditions. The significant issue in this, as in the case of any lossy resonator, is the definition of resonance. With regard to this question the viewpoint adopted here is that by resonance of a lossy system is meant the phenomenon that takes place when, under steady-state sinusoidal operating conditions, the response of the resonator reaches a relative maximum with variations in frequency. The corresponding frequencies are, by definition, the resonant frequencies of the resonator.

The next problem, therefore, is to determine the condition of resonance for the specimen-loaded cavity of Fig. 1.

It is a well-known fact that in a linear system the natural frequencies of oscillation are the poles of the transfer function for the particular problem being investigated or, stated in another way, the zeros of its denominator. Accordingly, if $D(s)$ denotes this denominator, the natural frequencies of oscillation are the roots of the algebraic equation $D(s)=0$ and the resonant frequencies may be defined by the roots of the equation

$$\frac{d}{d\omega}|D(j\omega)|=0.$$

By analogy, the *natural frequencies* of oscillation of the lossy, but linear, resonator of Fig. 1 are solutions of (3), while its *resonant frequencies* are solutions of the equation

$$\frac{d}{d\omega}(u^2+v^2)=0 \qquad (5)$$

where u and v are, respectively, the real and imaginary parts of the left-hand member of (3) evaluated at $s=j\omega$, $\gamma_0=j\beta_0$ and $\gamma=\alpha+j\beta$. The condition for resonance, expressed by (5), together with the relations $h^2+\beta_0^2=\omega^2\mu_0\epsilon_0$ and $h^2-\gamma^2=\omega\mu(\epsilon-j\sigma/\omega)$ constitute a set of four equations expressing relations among a total of six variables, namely β_0, α, β, ω, ϵ and σ. The desired quantities ϵ and σ (and, hence, the loss tangent) may be determined from these equations using measured values of either ω and β, or β_0 and β. (A method for measuring phase shift constants has been reported by Simmons.[5]) The tacit assumption is, of course, that the specimen-filled cavity must be at resonance and its dimensions must remain fixed while measurements of the selected pair of variables are being made. It is evident that while the results of these measurements could ultimately be used to evaluate the Q of the specimen, the solution of the problem at hand may be completed by the present method without introducing Q into the calculations.

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[5] A. J. Simmons, "TE_{01} Mode Components in the 3mm Region," presented at the Millimeter and Submillimeter Wave Conference, Orlando, Fla.; January 7-10, 1963.

Bounds on the Elements of the Susceptance Matrix for Asymmetrical Obstacles in Waveguides

There exists a method[1-3] for the determination of upper and lower bounds on the elements of the reactance matrix B, or the equivalent network elements, for multichannel scattering. This technique was applied[4] to specific examples of lossless obstacles in a rectangular waveguide, which are symmetric with respect to some plane perpendicular to the axis of the waveguide. The problem was analyzed in terms of uncoupled even and odd standing waves. Numerical results were obtained[3] for one-dimensional quantum mechanical scattering by an asymmetric potential $V(x)\neq V(-x)$.

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[2] L. Spruch and R. Bartram, "Bounds on the elements of the equivalent network for scattering in waveguides. I. Theory," *J. Appl. Phys.*, vol. 31, pp. 905–913; May, 1960.
[3] R. Bartram and L. Spruch, "Bounds on elements of the S matrix for elastic scattering· One-dimensional scattering," *J. Math Phys.*, vol. 3, pp. 287–296, March–April, 1961.
[4] R. Bartram and L. Spruch, "Bounds on the elements of the equivalent network for scattering in waveguides. II. Application to dielectric obstacles," *J. Appl. Phys.*, vol. 31, pp. 913–917; May, 1960.

It is the purpose of this communication to derive bounds on nonsymmetric obstacles in rectangular waveguide (see Fig. 1) by following the procedure of Bartram and Spruch,[3] and adapting certain of their results.[2-4] (We refer the reader to the above mentioned references for a discussion of the details which are only sketched or omitted here.)

The electric field intensity $\boldsymbol{E}(\boldsymbol{r})$ satisfies the differential matrix equation

$$\mathcal{L}\boldsymbol{E}=-\nabla\times\nabla\times\boldsymbol{E}+[(\omega^2/c^2)+\boldsymbol{V}]\boldsymbol{E}=0. \qquad (1)$$

$\boldsymbol{E}$ and the matrix potential $\boldsymbol{V}$ are expressed in terms of even and odd functions of z, the direction of propagation.

$$\boldsymbol{E}=\begin{pmatrix}\mathbf{E}_e\\ \mathbf{E}_0\end{pmatrix}, \qquad \boldsymbol{V}=\frac{1}{2}\begin{pmatrix}W_e & W_0\\ W_0 & W_e\end{pmatrix}, \qquad (2)$$

where

$$W_e=W_0=W=\omega^2(\epsilon-1)/c^2.$$

ω, c and ϵ are the angular frequency, velocity of light and relative permittivity of the obstacle, respectively. Since the two channels (corresponding to the even and odd portions of the electric field) are coupled by the matrix potential, three parameters are required to describe the asymptotic effects of the scattering process. The asymptotic form of $\boldsymbol{E}$ for $z\to+\infty$ is

$$\boldsymbol{E}=\mathbf{f}(x,y)[\boldsymbol{e}_\theta\cos(kz+\theta)-\boldsymbol{B}_\theta\boldsymbol{e}_\theta\sin(kz+\theta)], \qquad (3)$$

where $\mathbf{f}(x,y)$ is the form function for the propagating mode, $\boldsymbol{B}_\theta$ is the susceptance matrix, $\boldsymbol{e}_\theta$ is an amplitude column matrix $0\leq\theta\leq\pi$, and k^2 is $(\omega/c)^2-(\pi/a)^2$ (a is the wide dimension of the guide).

In order to obtain bounds on the susceptance matrix we have to consider an associated eigenvalue problem with certain boundary conditions,

$$\mathcal{L}\boldsymbol{\psi}_n(\mathbf{r})+\boldsymbol{\mu}_n\boldsymbol{\rho}(\mathbf{r})=0, \qquad (4)$$

where $\boldsymbol{\psi}_n$ and μ_n are its eigenfunctions and eigenvalues, respectively, and where $\rho(\mathbf{r})$ is a real, positive definite Hermitian matrix. Let α_θ and $-\beta_\theta$ be the smallest positive and smallest (in absolute value) negative eigenvalue, respectively, associated with the eigenmodes of (4). The upper and lower bounds on a quadratic form of the susceptance matrix are[5]

$$-\alpha_\theta^{-1}\int(\mathcal{L}\boldsymbol{E}_t)^\dagger(\boldsymbol{\rho}^{-1}\mathcal{L}\boldsymbol{E}_t)d\tau \leq k\boldsymbol{e}_\theta^\dagger\boldsymbol{B}_{\theta t}\boldsymbol{e}_\theta-k\boldsymbol{e}_\theta^\dagger\boldsymbol{B}_\theta\boldsymbol{e}_\theta+\int\boldsymbol{E}_t^\dagger\mathcal{L}\boldsymbol{E}_t d\tau \leq \beta_\theta^{-1}\int(\mathcal{L}\boldsymbol{E}_t)^\dagger(\boldsymbol{\rho}^{-1}\mathcal{L}\boldsymbol{E}_t)d\tau, \qquad (5)$$

where $\boldsymbol{E}_t$ is a trial function which is required to have the asymptotic form of $\boldsymbol{E}$, (3), but the unknown $\boldsymbol{B}_\theta$ is replaced by $\boldsymbol{B}_{\theta t}$. The range of integration of $d\tau$ is over the interior of the waveguide ($z\geq0$).

The above theory will now be applied to nonsymmetric obstacles in waveguide extending a distance d in the z direction (see

[5] The symbol † stands for the Hermitian adjoint.

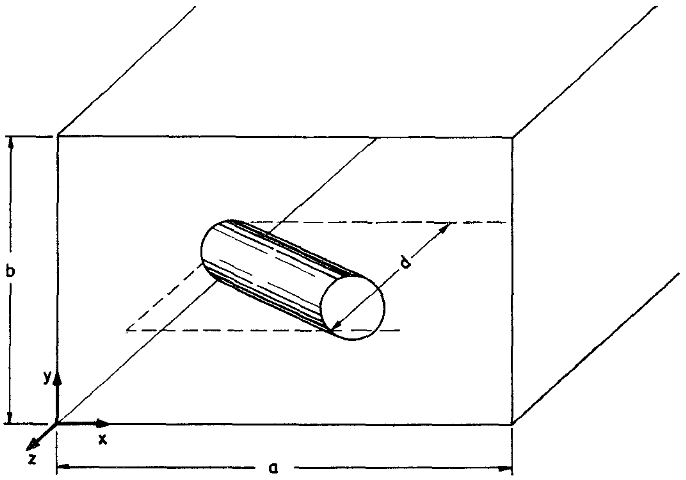

Fig. 1—A dielectric obstacle in a rectangular waveguide. The obstacle is nonsymmetric with respect to a plane perpendicular to the axis z of the waveguide. The cylinder extends a distance d in the z direction.

Fig. 1). In order to obtain bounds on the three distinct elements of B_θ, we evaluate (5) by using three different forms of the trial function, E_t. To do this we choose the following three values for e_θ,

$$e_\theta = \begin{pmatrix}1\\0\end{pmatrix},\ \begin{pmatrix}0\\1\end{pmatrix},\ \text{and}\ \begin{pmatrix}1\\1\end{pmatrix}. \qquad (6)$$

The corresponding values of $e_\theta^\dagger B_\theta e_\theta$ are then B_{11}, B_{22}, and $B_{11}+B_{22}+2B_{12}$. The exact solution of a dielectric slab, which fills the region $0 \leq z \leq d$ of the waveguide (the slab completely encloses the obstacle), is introduced as a trial function. The permittivity of the slab is retained as a parameter which can be varied to improve the bounds. The trial function within the region of the dielectric slab is then

$$E_t = \begin{pmatrix}\mathbf{j}C \sin(\pi x/a)\cos Kz\\ \mathbf{j}D\sin(\pi x/a)\sin Kz\end{pmatrix}, \qquad (7)$$

where K is the parameter to be varied, C and D are constants and $\mathbf{j}$ is a unit vector in the y direction. $e_\theta^\dagger B_{\theta t} e_\theta$, and C and D are determined by matching the tangential components E_t in (7) and H_t at $z=d$ to the asymptotic trial expressions (B_θ replaced by $B_{\theta t}$) of (3), and by specifying the value of θ. It can be shown that for

$$\theta = \pi - kd,\ d(k^2+W)^{1/2} < \frac{1}{2}\pi$$

we have

$$\beta_\theta \to -\infty$$

and

$$\alpha_\theta > [(\pi/2)^2 - k^2d^2 - Wd^2]/\rho d^2. \qquad (8)$$

The requirement $d(k^2+W)^{1/2} < \frac{1}{2}\pi$ means that the axial extent d of the obstacle must be *less* than $\frac{1}{4}\lambda_g$, where λ_g is the guide wavelength in the dielectric.

Substituting (6), (7) and (8) in (5), we obtain the upper and lower bounds on B_{11}, B_{22}, and $B_{11}+B_{22}+2B_{12}$:

$$-\sec^2(Kd)[P^2Q^+ + (2PR+R^2)I_e](\alpha_\theta')^{-1} \leq Kd\tan(Kd) - KdB_{11} + \sec^2(Kd)(PQ^+ + RI_e) \leq 0, \qquad (9a)$$

$$-\csc^2(Kd)[P^2Q^- + (2PR+R^2)I_0](\alpha_\theta')^{-1} \leq -Kd\cot Kd - KdB_{22} + \csc^2(Kd)(PQ^- + RI_0) \leq 0, \qquad (9b)$$

$$-\{\sec^2(Kd)[P^2Q^+ + (2PR+2R^2)I_e] + \csc^2(Kd)[P^2Q^- + (2PR+2R^2)I_0] + 4(PR+R^2)\sec(Kd)\csc(Kd)I\}(\alpha_\theta')^{-1} \leq Kd[\tan(Kd) - \cot(Kd)] - Kd(B_{11}+B_{22}+2B_{12}) + \sec^2(Kd)(PQ^+ + RI_e) + \csc^2(Kd)(PQ^- + RI_0) + 2R\sec(Kd)\csc(Kd)I \leq 0, \qquad (9c)$$

where

$$P \equiv (kd)^2 - (Kd)^2$$

$$Q^{+,-} \equiv \tfrac{1}{2}[1 \pm \sin(2Kd)/(2Kd)]$$

$$R \equiv \tfrac{1}{2}Wd^2$$

$$\alpha_\theta' \equiv (\pi/2)^2 - (k^2+W)d^2$$

$$I_e = (2/abd)\int_{\text{obst}} \sin^2(\pi x/a)\cos^2(Kz)d\tau$$

$$I_0 = (2/abd)\int_{\text{obst}} \sin^2(\pi x/a)\sin^2(Kz)d\tau$$

$$I = (2/abd)\int_{\text{obst}} \sin^2(\pi x/a)\cos(Kz)\sin(Kz)d\tau$$

(b is the narrow dimension of the waveguide). The range of integration in I_0, I_e and I is over the volume of the obstacle.

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Note on Tabulations of Constants for Rigid Hollow Metal Rectangular Waveguide

Precise four decimal place tables of free-space and waveguide wavelength and related ratios for rigid hollow metal rectangular waveguides were computed by the Sperry Microwave Electronics Company and published in this journal in 1956.[1] This set of tables was later extended to cover 28 American waveguide sizes and appeared in a handbook.[2] Booth has published a set of microwave data tables including waveguide wavelength to three decimal places for ten commonly used British rectangular waveguide sizes.[3]

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Unfortunately, two of these tabulations[1,2] use a "low" value for the speed of light: $c=299776$ mks and thus contain errors in their tabulated constants. Booth's tabulations use the presently accepted value of $c=299792.5$ km/sec[4] but cover only ten British waveguide sizes.

The errors entailed by assuming the older "low" value of c can best be explained by examples comparing the free-space and waveguide wavelengths computed using the "low" and "accepted" values of c. Let us first examine the error entailed in the computa-

frequency (Gc)	λ (cm) (using $c=299776$ km/sec	λ (cm) (using $c=299792.5$ km/sec
0.275	109.0094	109.0155
1.000	29.9776	29.9793
100.000	0.2998	0 2998

tion of free-space wavelength. Thus it is seen that for frequencies below 100 Gc/ errors in λ may occur in the third or fourth decimal place tabulated if the "low" value of c is used. In a similar fashion, errors can be observed in waveguide wavelength for a single waveguide size. Let us, for example, examine λ_g for the common 2.000×1.000 inch outside dimension waveguide (IEC R-48, British WG-12, American WR-187 and RG-49/U numbers):

frequency (Gc)	λ_g(cm) (using $c=299776$ km/sec	λ_g(cm) (using $c=299792.5$ km/sec
3.600	17 2416	17.2454
4.800	8.2815	8.2823
6 400	5.3822	5.3825

Errors in λ_g can be observed in the third decimal place tabulated if the "low" value of c is used.

In summary, then, presently available tabulations of rectangular waveguide constants are either slightly restricted in scope or are present numbers that are slightly in error for the most commonly used waveguide sizes for frequencies below 100 Gc.

There are presently available 38 standard rectangular waveguide sizes in approximately two-to-one dimension ratio catalogued according to the International Electrotechnical Commission, American and British systems. To the author's knowledge, no one complete cross referencing of identification systems[5] or complete set of tables

[1] Sperry Microwave Electronics Co., "Tables of constants for rectangular waveguides," IRE Trans. on Microwave Theory and Techniques," vol. MTT-4, 12 page supplement; July, 1956.

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